Practical efficient optimization over large-scale subtour and semi-metric polytopes for sparse graphs

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1 Introduction

Given an undirected graph $G = (V, E)$ where $n = |V|$ and $m = |E|$, we say that G is sparse when $m = O(n)$. The subtour polytope $SP(G)$ is defined by the following linear system:

$$x(\delta(v)) = 2 \text{ for all } v \in V \tag{1}$$

$$x(\delta(S)) \geq 2 \qquad \text{for all } S \subset V \tag{2}$$

$$x \in [0, 1]^E \tag{3}$$

where $\delta(S) = \{uv \in E \mid \text{such that } u \in S \text{ and } v \in V \setminus S\}$ for all vertex subset $S \subset V$. The inequalities (2) are called *subtour inequalities*.

The subtour polytope plays an important role in combinatorial optimization since it represents the subtour elimination constraints for many problems involving tours (i.e closed paths) as solutions such as the Traveling Salesman Problem (TSP), the Vehicle Routing Problem (VRP) and its variants. In particular, the subtour polytope is the linear programming Held-Karp relaxation of the Traveling Salesman Problem (TSP).

The semi-metric polytope is defined by the following linear system:

$$x(F) - x(C \setminus F) \geq |F| - 1 \quad \text{for all cycles } C \text{ in } G \text{ and for all } F \subset C,\ |F| \text{ odd} \tag{4}$$

$$x \in [0, 1]^E \tag{5}$$

The semi-metric polytope defined by Barahona and Mahjoub [1986] is also an important relaxation for MaxCut problem and its variants, also for Graph Partitioning problem and its variants since the inequalities (4) called *cycle inequalities* express the partition of the vertex set into clusters. In particular, the semi-metric polytope is the linear programming relaxation of the Max-Cut problem.

Although the number of subtour inequalities (2) (respectively cycle inequalities (4)) is exponential, optimizing over the subtour polytope (respectively the cycle polytope) can be theoretically done in polynomial time by the ellipsoid algorithm as the separation problem of (2) (respectively (4)) can be solved in polynomial time. By their fundamental importance, the subtour and the semi-metric polytope have been so far subject of considerable amounts of works in the literature. One can cite the many efforts to prove the conjecture of the ratio 4/3 between the optimal solution for TSP over the one for the subtour polytope. The semi-metric polytope has been also intensively studied, especially on its extended formulation containing the triangle inequalities. Beside of theoretical works, practical previous works focus on how to efficiently handle the subtour and the cycle inequalities in practical solutions for TSP, VRP and variants, graph partitioning problem and variants. This is a very important since for example a great part of solution time for VRP using two index flow formulations is spent for handling the subtour elimination constraints. Nevertheless, there are very few works studying the optimization over the subtour and the semi-metric polytopes. We are aware of only the work of Boyd and Cunningham (Math. Prog 1990) on the subtour polytope and the work of Frangioni et al. (Math. Prog. 2015) on the semi-metric polytope. While the first is rather theoretical, the second reports that optimizing over the semi-metric could be very time-consuming and hence scalability is difficult. However, an efficient algorithm optimizing over the two polytopes can improve exact solutions for TSP, VRP, MaxCut, Graph Partionning problems ... since it treats the constraints among the most important and the most fundamental constraints in these problems.

2 Objective of the PhD thesis

The main objective of the PhD thesis is to propose efficient algorithms for optimizing over the subtour and the semi-metric polytopes for sparse graphs. The first phase of the thesis is the designing of algorithms.

- One of the direction in this phase is to improve the works of (Boyd and Cunningham) and (Frangioni et al) by using the sparsity of the graph.
- Other direction is to replace the exponential number of subtour and cycle inequalities in the models by some linear size formulation (which is not necessarily linear) of their separation problem. The sparsity of the graph should be exploited to keep the numbers of additional variables and inequalities are linear in term of n.

In the second phrase of PhD thesis, the PhD candidate will perform numerical experiments for proving the efficiency of his algorithms. The last phase will be dedicated to integrate the algorithms into existing approaches for improving the exact solutions for TSP, VRP, MaxCut, Graph Partionning problems.

3 Contribution of the thesis to the project I-Site CAP 20-25

One of the objectives of the thesis is to solve exactly large-scale instances (thousands vertices) of the subtour and semi-metric relaxations in a reasonable time. Hence it can be use to provide good lower bounds for large-scale instances of VRP and its variants. These lower bounds could be used to assert the quality of the heuristic solutions or be rounded in a rounding scheme to give good quality heuristic solutions.

4 Risk analysis

Although the proposal is about fundamental and well-studied problems such as TSP, VRP, MaxCut, the main focus is on a subproblem (optimizing over relaxations) which has not much attention so far. Moreover, preliminary results on the second direction are very promising (Viet Hung Nguyen and Minoux 2020, submitted). A good student is needed.