Enumeration Problems in Graphs, Hypergraphs, and Lattice Structures

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Enumeration problems

Typical question:

"Given input I, list all objects of type X in I."

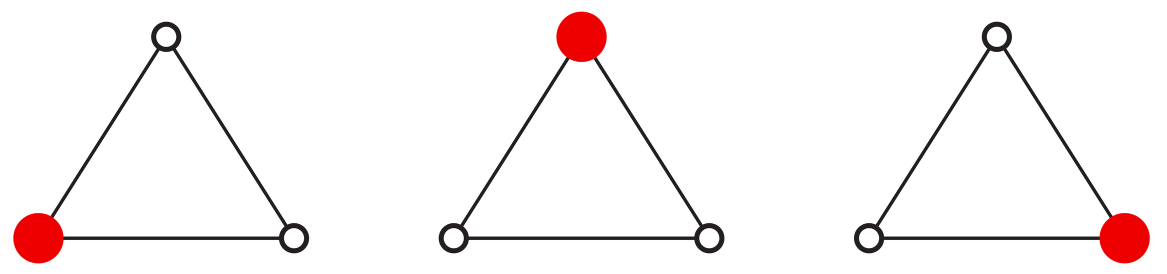
- cycles, cliques, independent sets, dominating sets, etc. of a graph
- transversals of a hypergraph
- elements of the (dual) antichain of a lattice
- variable assignments satisfying a formula
- trains to Paris leaving tomorrow before 10:00
- shortest paths from Montpellier to Marseille, etc.

Applications

Mainly in database theory (query answering, repairing), AI, bioinformatics.

Possibly many objects: *an example on graphs*

Definition: Let $G = (V, E)$ be a graph.
A subset $D \subseteq V$ of vertices of G is a dominating set if every vertex of G is either in D, or adjacent to a vertex of D. It is called minimal if it is minimal by inclusion, i.e., no vertex can be removed while remaining a dominating set.



Observation: there are graphs with $3^{n/3} \approx 1.44^n$ minimal dominating sets, where n is the number of vertices (best known upper bound is 1.71^n).

Complexity measures

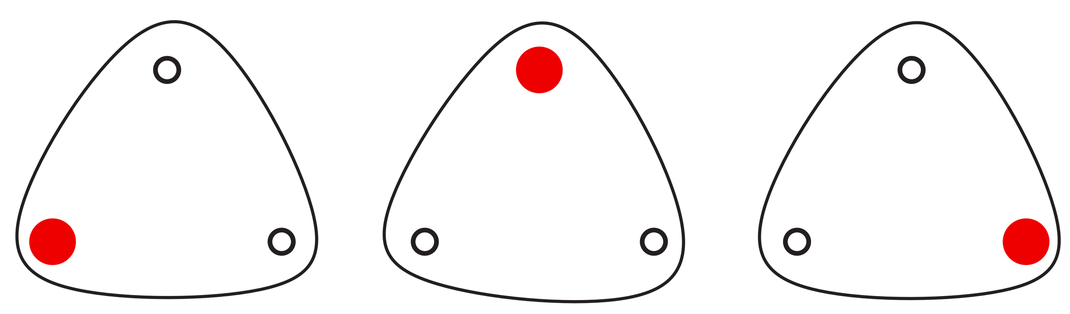
Input-sensitive: in terms of input size
There is an algorithm enumerating the minimal dominating sets of a n-vertex graph in $O(1.71^n)$ time. [Fomin, Grandoni and Pyatkin, 2008]

Output-sensitive: in terms of input+output size
There is an algorithm enumerating the minimal dominating sets of a n-vertex graph in $O(N^{\log N})$ time, where $N = n + d$ and d is the number of minimal dominating sets of G. [Fredman and Khachiyan, 1996]

Polynomial equivalence: two enumeration problems Π_1 and Π_2 are polynomially equivalent if there is an output-polynomial time algorithm solving Π_1 if and only if there is one solving Π_2.

A *transversal* problem

Definition: Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph.
A subset $T \subseteq V$ of vertices of $\mathcal{H}$ is a transversal if it intersects every hyperedge of $\mathcal{H}$. It is called minimal if it is minimal by inclusion, i.e., no vertex can be removed while remaining a transversal.



Observation: enumerating the minimal dominating sets of a graph amounts to enumerate the minimal transversals of its hypergraph of (min.) neighborhoods. In fact, both problems are equivalent [Kanté et al., 2014].

Open problems

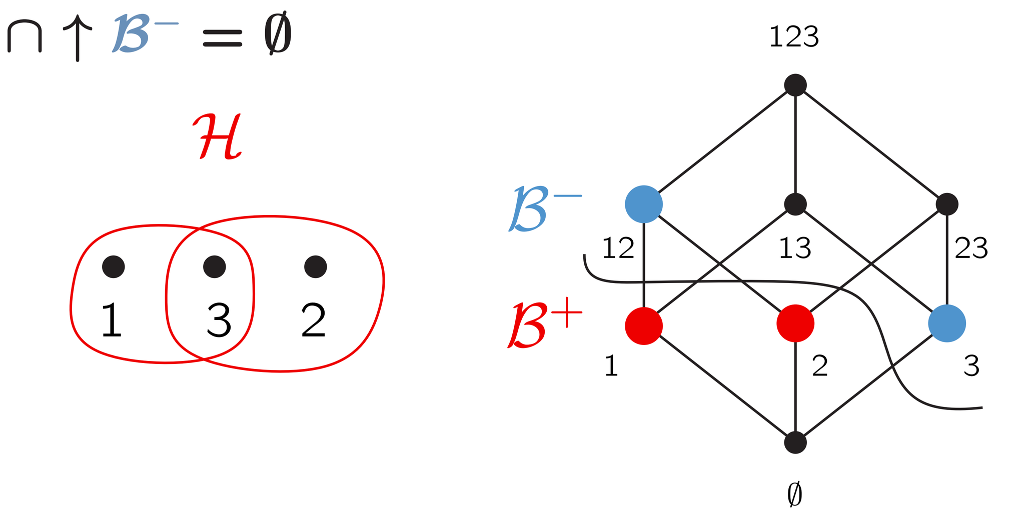
Is there an output-polynomial algorithm for Trans-Enum/Dom-Enum?

- what if the graph is bipartite? output-poly even for K_t-free! [1, 2].
- has no cycle of size four?
- enumeration with linear delay? in P_7-free chordal graphs! [3].

Dualization in Boolean lattices

Definitions: Let $P = (X, \leq)$ be a poset.
The ideal of x in P is $\downarrow x = \{y \mid y \leq x\}$. The filter of x in P is the dual $\uparrow x = \{y \in X \mid x \leq y\}$. If $S \subseteq X$ then $\downarrow S = \bigcup_{x\in S} \downarrow x$ and $\uparrow S = \bigcup_{x\in S} \uparrow x$. An antichain of P is a subset of elements that are not comparable. Two antichains $\mathcal{B}^+$ and $\mathcal{B}^-$ are dual if:

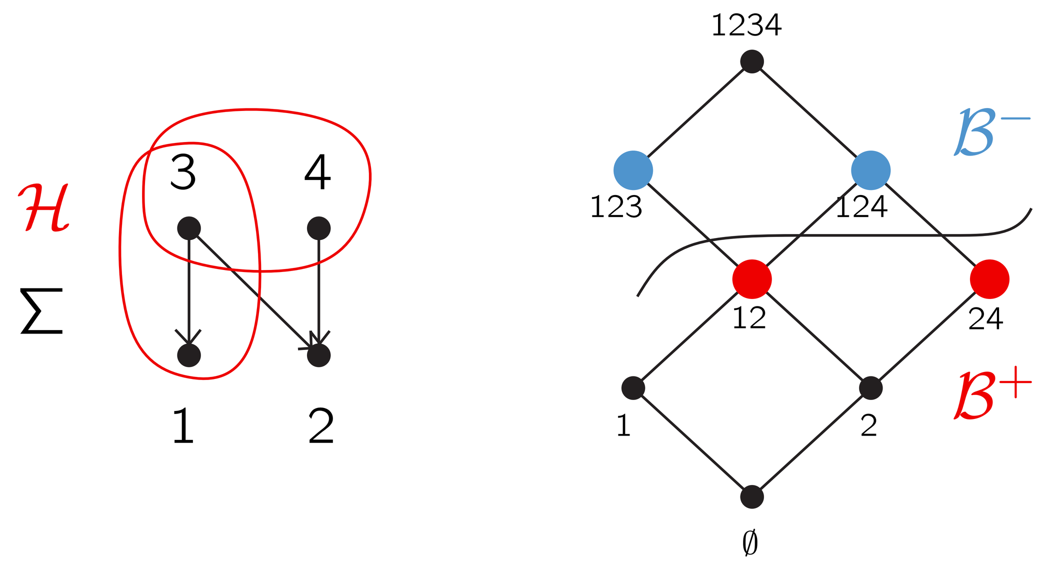
- $\downarrow \mathcal{B}^+ \cup \uparrow \mathcal{B}^- = X$
- $\downarrow \mathcal{B}^+ \cap \uparrow \mathcal{B}^- = \emptyset$



Observation: enumerating the minimal transversals of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ amounts to compute the dual antichain of its complementary $\mathcal{B}^+ = \{V \setminus E \mid E \in \mathcal{E}\}$ in the Boolean lattice obtained by ordering all subsets of V by inclusion, i.e., $P = (X, \subseteq)$ where $X = 2^V$.

Generalization to other lattices?

When generalized to other lattices, the lattice can be given by an implicational base Σ and consists of all sets that are closed according to the implications in Σ. The lattices is distributive if implications are of size one. In the figure below, 14 does not belong to the lattice as it is not closed in Σ.



Observation: given the antichain $\mathcal{B}^+$ of a distributive lattice, computing its dual antichain $\mathcal{B}^-$ amounts to compute the minimal closed (in Σ) transversals of its complementary $\mathcal{B}^+ = \{V \setminus E \mid E \in \mathcal{E}\}$.

Complexity status

Distributive lattices: open in general. The problem is equivalent to the Boolean case if the graph of implications is of bounded induced matching [4]. Output-polynomial are known under various restriction on the antichain and the implicational base coding the lattice [5].

General lattices: the dualization is impossible in output-polynomial time unless $P = NP$, even when the implicational base coding the lattice is of dimension two [4].

References

[1] Marthe Bonamy, Oscar Defrain, Marc Heinrich, and Jean-Florent Raymond.
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[2] Marthe Bonamy, Oscar Defrain, Marc Heinrich, Michał Pilipczuk, and Jean-Florent Raymond.
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arXiv preprint arXiv:1810.00789, 2019.

[3] Oscar Defrain and Lhouari Nourine.
Neighborhood preferences for minimal dominating sets enumeration.
arXiv preprint arXiv:1805.02412, 2018.

[4] Oscar Defrain and Lhouari Nourine.
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[5] Oscar Defrain, Lhouari Nourine, and Takeaki Uno.
On the dualization in distributive lattices and related problems.
arXiv preprint arXiv:1902.07004, 2019.