Computational and Digital Geometry

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Introduction

Digital geometry is the field of mathematics that studies the geometry of points with integer coordinates, also known as *lattice points* [1]. Convexity is a fundamental concept in digital geometry, as well as in continuous geometry [2]. From a historical perspective, the study of digital convexity dates back to the works of Minkowski [3] and it is the main subject of the mathematical field of geometry of numbers. While convexity has a unique well stated definition in any linear space, different definitions have been investigated in $\mathbb{Z}^2$ and $\mathbb{Z}^3$. In two dimensions, we encounter at least five different approaches, called respectively digital line, triangle, HV (for Horizontal and Vertical), and Q (for Quadrant) convexities. These definitions were created in order to guarantee that a digital convex set is connected (in terms of the induced grid subgraph), which simplifies several algorithmic problems.

Definition

The original definition of digital convexity in the geometry of number does not guarantee connectivity of the grid subgraph, but provides several other important mathematical properties, such as being preserved under certain affine transformations (Fig. 1). The definition is the following.

digital convex: A set of lattice points $S \subset \mathbb{Z}^d$ is *digital convex* if $\mathbf{conv}(S) \cap \mathbb{Z}^d = S$, where $\mathbf{conv}(S)$ denotes the convex hull of S.

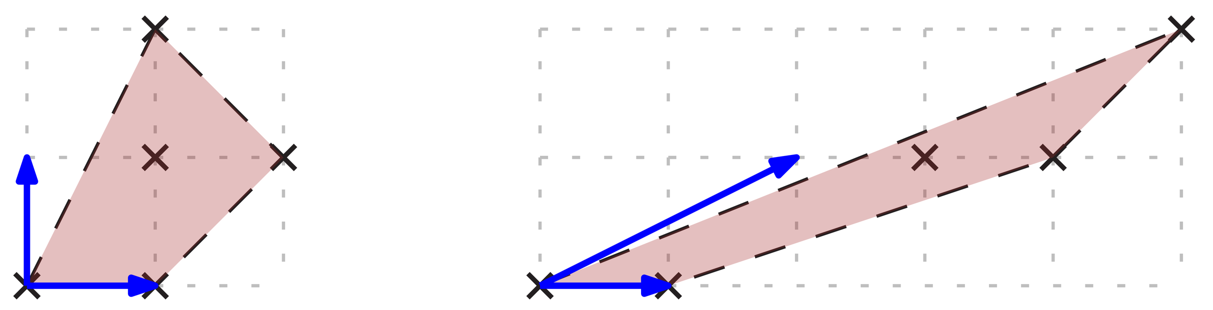

Figure 1: **Shearing a digital convex set.** Example of a set whose connectivity is lost after a linear shear.

Problems considered

Here are a few if the problems considered: Given set of lattice:

- Is the set digital convex ?

  Problem `Test Convexity`(S)
  Input: Set $S \subset \mathbb{Z}^d$ of n lattice points given by their coordinates.
  Output: Determine whether S is digital convex or not.

- What is the largest digital convex subset ?

  Problem `Digital Potato Peeling` (S)
  Input: Set $S \subset \mathbb{Z}^2$ of n lattice points given by their coordinates.
  Output: Determine the *largest* set $K \subseteq S$ that is digital convex (i.e., $\mathbf{conv}(K) \cap \mathbb{Z}^2 = K$), where largest refers to the area of $\mathbf{conv}(K)$.

- What is the largest union of k digital convex subsets ?

  Problem `Digital k-Potato Peeling` (S)
  Input: Set $S \subset \mathbb{Z}^2$ of n lattice points given by their coordinates.
  Output: Determine the *largest* union of sets $K = K_1 \cup K_2 \cup \ldots \cup K_k$ such that $\forall i \in [1..k] : K_i \subseteq S$ is digital convex.

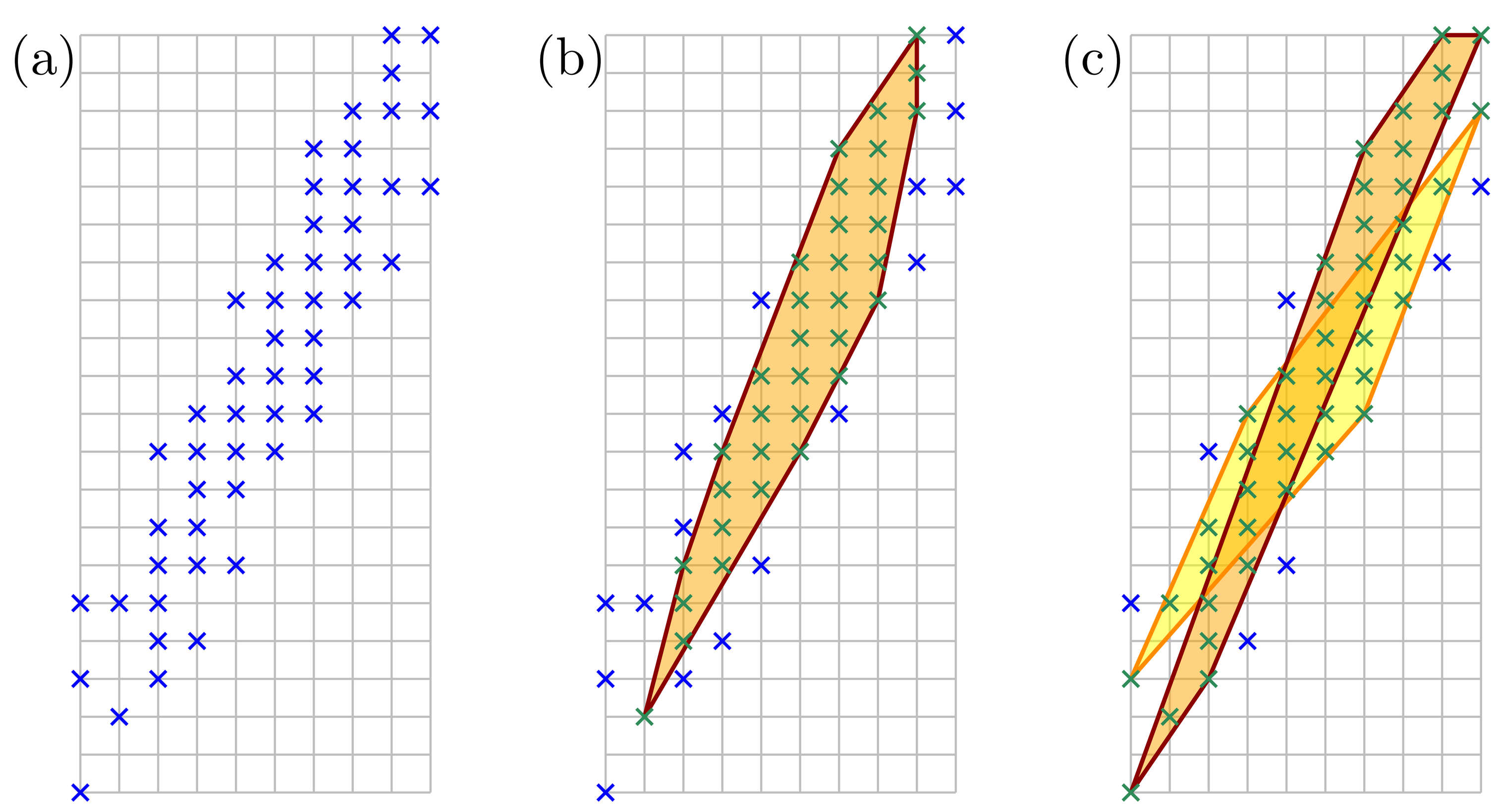

Figure 2: (a) Input lattice set S. (b) Largest digital convex subset of S. (c) Largest union of two digital convex subsets of S.

Results and Conclusion

Many problems in the intersection of digital, convex, and computational geometry remain open. Our study falls in the following framework of problems, all of which receive as input a set of n lattice points $S \subset \mathbb{Z}^d$ for constant d and are based on a fixed parameter $k \geq 1$.

1. Is S the union of at most k digital convex sets?
2. What is the smallest superset of S that is the union of at most k digital convex sets?
3. What is the largest subset of S that is the union of at most k digital convex sets?

In [4] we considered the first problem for $k = 1$, presenting polynomial time solutions (which may still leave room for major improvements for $d > 3$). We are not aware of any previous solutions for $k > 1$. In contrast, the continuous version of the problem is well studied. The case of $k = 1$ can be solved easily by a convex hull computation or by linear programming. Polynomial algorithms are known for $d = 2$ and $k \leq 3$ [5, 6], as well as for $d = 3$ and $k \leq 2$ [7]. The problem is already NP-complete for $d = k = 3$ [7]. Hence, the continuous version remains open only for $d = 2$ and fixed $k > 3$.

It is easy to obtain polynomial time algorithms for the second problem when $k = 1$, since the solution consists of all points in the convex hull of S. The continuous version for $d = k = 2$ can be solved in $O(n^4 \log n)$ time [8]. Also, the orthogonal version of the problem is well studied (see for example [9]). We know of no results for the digital version.

In a yet to be published paper, we considered the digital version of the third problem for $d = 2$ and $k = 1, 2$, proposing algorithms with respective running times of $O(n^3 + n^2 \log r)$ and $O(n^9 + n^6 r)$, where r is the diameter of S. Since the first problem trivially reduces to the third problem, we also solved the first problem for $k = d = 2$ in $O(n^9 + n^6 r)$ time. It is surprising that we are not aware of any faster algorithm for the first problem in this particular case.

The third problem for $d > 2$ or $k > 2$ remains open. Surprisingly, even the continuous version seems to be unresolved for $d > 2$ or $k \geq 2$.

References

[1] Reinhard Klette and Azriel Rosenfeld.
Digital geometry: Geometric methods for digital picture analysis.
Elsevier, 2004.

[2] Christian Ronse.
A bibliography on digital and computational convexity (1961-1988).
IEEE Transactions on Pattern Analysis and Machine Intelligence, 11(2):181–190, February 1989.

[3] H. Minkowski.
Geometrie der Zahlen.
Number vol. 2 in Geometrie der Zahlen. B.G. Teubner, 1910.

[4] Loïc Crombez, Guilherme D. da Fonseca, and Yan Gérard.
Efficient algorithms to test digital convexity.
In Michel Couprie, Jean Cousty, Yukiko Kenmochi, and Nabil Mustafa, editors, *Discrete Geometry for Computer Imagery*, pages 409–419, Cham, 2019. Springer International Publishing.

[5] Patrice Belleville.
On restricted boundary covers and convex three-covers.
In *5th Canadian Conference on Computational Geometry (CCCG)*, pages 467–472, 1993.

[6] Thomas C. Shermer.
On recognizing unions of two convex polygons and related problems.
Pattern Recognition Letters , 14(9), pages 737–745, 1993.

[7] Patrice Belleville.
Convex covers in higher dimensions.
In *7th Canadian Conference on Computional Geometry (CCCG)*, pages 145–150, 1995.

[8] Sang Won Bae, Hwan-Gue Cho, William Evans, Noushin Saeedi, and Chan-Su Shin.
Covering points with convex sets of minimum size.
Theoretical Computer Science, 718:14–23, 2018.

[9] Cem Evrendilek, Burkay Genç, and Brahim Hnich.
Covering points with minimum/maximum area orthogonally convex polygons.
Computational Geometry, 54:32–44, 2016.

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