The dichotomy and the description questions for complexity classes

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General objectives

1. **Dichotomy.** To find the necessary and the sufficient conditions for a complexity class to have dichotomy.
2. **Description.** To find logical languages that describe certain complexity classes.

P and NP classes

- **P.** A problem belongs to the class P if there exists an algorithm that solves it in the time polynomial of the size of the input of the problem.
- **NP.** A problem belongs to the class NP if there exists an algorithm that checks if the proposed solution to the problem is correct in the time polynomial of the size of the input of the problem.

Logical description

Let $\mathfrak{L}$ be a logic i.e. a set of logical sentences; and let $\mathfrak{C}$ be a complexity class i.e. a set of problems. We say that $\mathfrak{L}$ *describes* $\mathfrak{C}$ if there is a two-sided polynomial time reduction φ_1, φ_2 from $\mathfrak{L}$ to $\mathfrak{C}$ i.e. the following hold:

- For every sentence $\Phi \in \mathfrak{L}$ it is possible to find in polynomial time a problem $\mathcal{A} \in \mathfrak{C}$ such that for all graphs G:
$$G \models \Phi \Leftrightarrow \varphi_1(G) \in \mathcal{A}$$
- For every problem $\mathcal{A} \in \mathfrak{C}$ it is possible to find in polynomial time a sentence $\Phi \in \mathfrak{L}$ such that for all graphs G:
$$G \models \Phi \Leftrightarrow \varphi_2(G) \in \mathcal{A}$$

MMSNP logic

The class *MMSNP* consists of formulas of the form:
$$\exists P_1, \dots, P_n \forall x_1, \dots, x_m \bigwedge_i \neg(\alpha_i \wedge \beta_i)$$
where

- α_i is a conjunction of atoms of the form $E(x_i, x_j)$ where $E(\cdot,\cdot)$ is an input edge relation.
- β_i is a conjunction of atoms of the form $P_i(x_j)$ or $\neg P_i(x_j)$ where $P_i(\cdot)$ is a unary existentially defined relation on the graph vertices.

Known results

- The class *NP* doesn't have dichotomy. See [Lad75].
- The class *CSP* has dichotomy. See [Zhu17] and [Bul17].
- The logic *MMSNP* describes the class *CSP*. See [FV99].

MMSNP* as a candidate for M-partition

The class *MMSNP** consists of formulas of the same form as *MMSNP*:
$$\exists P_1, \dots, P_n \forall x_1, \dots, x_m \bigwedge_i \neg(\alpha_i \wedge \beta_i)$$
with the only difference: α_i is either a conjunction of atoms $E(x_i, x_j)$ or a conjunction of negated atoms $\neg E(x_i, x_j)$.

The sketch of possible proof is below:

$$MMSNP^* \longleftrightarrow \mathfrak{F}_1 \longleftrightarrow \mathfrak{F}_2 \longleftrightarrow M\text{-partition}$$

Here:

- Each arrow stands for two-sided polynomial time reduction.
- $\mathfrak{F}_1$ – the class of families of graphs such that for every $\varphi \in MMSNP^*$ there exists a family $\mathcal{F}_\varphi \in \mathfrak{F}_1$ such that the set $\mathcal{L}_\varphi = \{G - \text{graph} : G \models \varphi\}$ coincides with the set $Forb(\mathcal{F}_\varphi) = \{G - \text{graph} : \forall G' \in \mathcal{F}_\varphi . G' \not\to G\}$.
- $\mathfrak{F}_2$ – the class of families of graphs such that for every problem $\mathcal{M}$ from M*-partition* there exists a family $\mathcal{F}_\mathcal{M} \in \mathfrak{F}_2$ such that the set $\mathcal{L}_\mathcal{M} = \{G - \text{graph} : G \text{ satisfies } \mathcal{M}\text{-partition}\}$ coincides with the set $Forb(\mathcal{F}_\mathcal{M}) = \{G - \text{graph} : \forall G' \in \mathcal{F}_\mathcal{M} . G' \not\to G\}$.

This approach was used to show that *MMSNP* describes *CSP* in [KN08].

Particular objectives

1. **Dichotomy.** To check if the complexity class *M-partition* has dichotomy or not.
2. **Description.** To find a logical language that describes the complexity class *M-partition*.

P-time reduction and NP-completeness

- **P-time reduction.** A problem $\mathcal{A}$ is said to be *reducible* to a problem $\mathcal{B}$ if there is a polynomial time algorithm φ that maps instances of the problem $\mathcal{A}$ to instances of $\mathcal{B}$ such that for every instance A of $\mathcal{A}$ the following is true:
$$A \text{ is true in } \mathcal{A} \Leftrightarrow \varphi(A) \text{ is true in } \mathcal{B}$$
- **NP-completeness.** A problem $\mathcal{C}$ belongs to the class *NP-complete* if the following conditions hold:
  - $\mathcal{C}$ belongs to the class *NP*.
  - Every problem from *NP* is reducible to $\mathcal{C}$.

Dichotomy

Let $\mathfrak{C}$ be a complexity class. We say that $\mathfrak{C}$ *has dichotomy* if
$$\mathfrak{C} \subseteq P \sqcup NP\text{-complete}$$
We assume that $P \neq NP$ because otherwise this concept has no sense.

CSP class

Let G and H be graphs. We say that a map $h : G \to H$ is *a homomorphism between* G *and* H if for every $u, v \in G$:
$$(u, v) \text{ is an edge of } G \Rightarrow (h(u), h(v)) \text{ is an edge of } H.$$
Let H be fixed. The problem that decides if there is a homomorphism from the input G to H is denoted by *CSP(H)*.

M-partition

Let M be an $n \times n$ matrix with elements from $\{0, 1, *\}$. We say that a graph G *satisfies* M*-partition* if there is a partition of its vertices into n classes $M_1, \dots, M_n$ such that the following is preserved:

- if an element m_{ij} of M equals 0 then for any $u \in M_i$ and $v \in M_j$ there is no edge (u, v) in G;
- if $m_{ij} = 1$ then for any $u \in M_i$ and $v \in M_j$ (u, v) is an edge of G;
- if $m_{ij} = *$ then there is no restriction for any $u \in M_i$ and $v \in M_j$.

Let matrix M be fixed. The problem that decides if a given as an input graph G satisfies M-partition is called an M*-partition problem*.

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